Journal of Pure and Applied Mathematics: Advances and Applications
Volume 8, Number 1, 2012, Pages 25-40

ANALYSIS OF THE EFFECTS OF LARGE SCALE MARINE IRON FERTILIZATION

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Abstract

A nonlinear integro-differential equation is used to investigate the effects of ocean iron fertilization on the evolution of the phytoplankton biomass. This equation contains terms responsible for fragmentation, coalescence, growth-decay, grazing, and sinking of the phytoplankton aggregates. The evolution equation is analyzed by using the theory of semilinear dynamical systems and numerical simulations are performed. Our results demonstrate the validity of the iron hypothesis in fighting climate change.

1. Introduction

Phytoplankton are microscopic plant-like marine organisms that sit at the bottom of the food chain. They are food for other plankton and small fish, as well as larger animals such as whales. Phytoplankton get

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2010 Mathematics Subject Classification: 34K30, 35R09, 35R10, 45K05, 47G20.

Keywords and phrases: phytoplankton, iron fertilization, semigroup, coagulation, fragmentation.

The research was partially supported by the National Research Foundation of South Africa under grant UID 78806.

Received August 30, 2012

their energy from carbon dioxide through photosynthesis and so are very important in carbon cycling. As each year, they transfer billion tonnes of carbon from the atmosphere to the ocean reducing global warming in the process, they are of primary interest to oceanographers and earth scientists around the world. These tiny marine organisms, which are crucial components of marine ecosystems, have been slowly disappearing over the last century, according to researchers [10, 12]. The decline is worrying because it may have profound effects on marine life and climate change. The major decrease has been recorded in the High Nitrate Low Chlorophyll (HNLC) regions that are thought to represent about 20 percent of the areal extent of the world's oceans ([10] and references therein). These are generally regions characterized by more than 2 micromolar nitrate and less than 0.5 micrograms chlorophyll per liter. The major HNLC regions include the Subarctic Pacific, large regions of the Eastern Equatorial Pacific and the Southern ocean. These HNLC regions persist in areas which have high macro-nutrient concentrations, adequate light, and physical characteristics required for phytoplankton growth, but have very low plant biomass. It is believed that phytoplankton growth in major nutrient-rich HNLC regions is limited by iron deficiency [10, 12]. The main purpose of this paper is to show that global warming can be substantially reduced and to some extent annihilated by fertilizing the HNLC areas of the oceans by a very modest amount of iron. The formation of large particles (aggregates) through multiple collision of smaller ones is a highly visible phenomenon in oceanic waters. Several authors have attempted to model the dynamics of phytoplankton in such a way as to exhibit this structure [1, 2, 4, 5, 7, 8, 9, 11, 13, 15, 17]. In this setting, the individual unit is an aggregate and aggregates are structured by their mass. One of the most efficient approaches to modelling the dynamics of phytoplankton aggregates is through a rate equation, which describes the evolution of the distribution of interacting aggregates with respect to their mass. The evolution equation contains terms responsible for the coalescence, disaggregation, growth-decay, sinking to the seabed of the aggregates and their grazing by the zooplankton. The novelty in our model from a mathematical point of view is that, we allow the kernels to vary according to the level of marine iron concentration. Next, we present a full description of the phytoplankton aggregates model used in this article and provide the assumptions. Then, we make use of the theory of semilinear abstract Cauchy problem used to analyze coagulation fragmentation processes with growth [2, 3, 4, 5, 9, 13, 14, 15, 17] or decay [3, 15] in order to show the well-posedness of the adopted model. In the last part of the article, numerical simulations are performed and the results are discussed.

2. Description of the Model and Assumptions

Following [4], we consider the following nonlinear transport equation that contains terms responsible for the growth/decay of phytoplankton aggregates, their fragmentation, coagulation, grazing, and sinking of aggregates into the seabed:

$$\begin{aligned}\frac{\partial}{\partial t} u_{\zeta}(t, x) = &-\frac{\partial}{\partial x}[r_{\zeta}(x)u_{\zeta}(t, x)] - s_{\zeta}(x)u_{\zeta}(t, x) - d(x)u_{\zeta}(t, x)\\ &- a(x)u_{\zeta}(t, x) + \int_x^{\infty} a(y)b(x|y)u_{\zeta}(t, y)dy\\ &- u_{\zeta}(t, x)\int_0^{\infty} k(x, y)u_{\zeta}(t, y)dy\\ &+ \frac{1}{2}\int_0^x k(x-y, y)u_{\zeta}(t, x-y)u_{\zeta}(t, y)dy, \qquad (2.1)\end{aligned}$$

where ζ represents the iron concentration in the sea. The sinking rate and the growth-decay rate of the clusters are denoted by s_{ζ} and r_{ζ}, respectively, they are ζ-dependent. Here $x \in \mathbb{R}_+$ represents the size of particles, t is the time variable, and u_{ζ} is the density of particles of mass x. The fragmentation rate is denoted by a and b describes the distribution of masses x of particles spawned by the fragmentation of a particle of mass y. The removal of phytoplankton aggregate is carried out by the grazing of the population by the zooplankton and the clusters sinking in to the seabed. The grazing rate is denoted by $d(x)$ and it is assumed that

$$d \in L_\infty(\mathbb{R}_+). \tag{2.2}$$

We introduce the following notation for formal expressions appearing in (2.1):

$$[\mathcal{T}_\zeta u_\zeta](x) = -\frac{\partial}{\partial x}[r_\zeta(x)u_\zeta(x)] - q_\zeta(x)u_\zeta(x), \tag{2.3}$$

$$[\mathcal{B}u_\zeta](x) = \int_x^\infty a(y)b(x|y)u_\zeta(y)dy, \tag{2.4}$$

$$[\mathcal{K}u_\zeta](x) = -u_\zeta(x)\int_0^\infty k(x, y)u_\zeta(y)dy$$

$$+\frac{1}{2}\int_0^x k(x-y, y)u_\zeta(x-y)u_\zeta(y)dy, \tag{2.5}$$

where $q_\zeta = a + d + s_\zeta$.

2.1. Assumptions on the coefficients

The sinking function $s_\zeta \geq 0$ represents the removal rate of the aggregates of phytoplankton into the seabed; it is assumed that for any fixed $\zeta \in \mathbb{R}_+$,

$$s_\zeta \in L_\infty(\mathbb{R}_+). \tag{2.6}$$

We assume that the fragmentation rate a is essentially bounded on compact subintervals of $\overline{\mathbb{R}}_+$; i.e.,

$$a \in L_{\infty, loc}(\overline{\mathbb{R}}_+). \tag{2.7}$$

Further, $b \geq 0$ is assumed to be a measurable function of two variables, satisfying

$$b(x|y) = 0; \quad \text{for} \quad x > y. \tag{2.8}$$

The local law of mass conservation requires

$$\int_0^y xb(x|y)dx = y, \quad \text{for each} \quad y > 0. \tag{2.9}$$

The coagulation kernel $k(x, y)$ represents the likelihood of a particle of size x attaching itself to a particle of size y and we assume

$$0 \leq k \in L_\infty(\mathbb{R}_+^2). \tag{2.10}$$

The transport part is more tortuous. Our principal assumption is that clusters of phytoplankton grow $(r_\zeta > 0)$ when the iron concentration ζ in the sea is bigger than a critical value ζ_c and they decay otherwise $(r_\zeta < 0)$. In phytoplankton models typically, we have $r_\zeta(x) \sim x$ as growth/decay is proportional to number of particles (cells) in the aggregate. Thus, we assume that

$$|r_\zeta(x)| \leq \tilde{r}x, \tag{2.11}$$

for some constant $\tilde{r} > 0$ and

$$r_\zeta \in AC(\mathbb{R}_+), \tag{2.12}$$

where $r_\zeta \in AC(\mathbb{R}_+)$ means that r_ζ is absolutely continuous on each compact subinterval of $\mathbb{R}_+$. Further assumptions on r_ζ depend on whether we deal with the decay or growth case. As we shall see, in the decay case, there is no need for boundary conditions. On the other hand, depending on the integrability of r_ζ at $x = 0$, the transport equation describing growth may require a boundary condition at $x = 0$. In this paper, we consider the general McKendrick-von Foerster renewal boundary condition

$$\lim_{x \to 0^+} r_\zeta(x)u_\zeta(t, x) = \int_0^\infty \beta_\zeta(y)u_\zeta(t, y)dy, \tag{2.13}$$

where β_ζ is a suitable positive measurable function for any $\zeta \neq 0$. If $\beta_\zeta \equiv 0$, then we have standard no-influx condition. If, however, $\beta_\zeta(y) > 0$, then it describes the rate at which an aggregate of size y sheds monomers of the smallest 'zero' size, which then re-enter the system as new aggregates and start to grow. The nonlinear integro-differential equation (2.1) will be supplemented with an initial condition.

3. Analysis of the Problem

Our approach is to analyze the evolution equation in the Banach space

$$X_{0,1} := L_1((0,\infty), (1+x)dx) = \left\{\phi;\ \int_0^\infty |\phi(x)|(1+x)dx < +\infty\right\}, \tag{3.1}$$

in which both the total mass and the number of particles are controlled. In order to ensure the validity of the general McKendrick-von Foerster renewal boundary condition, we further assume that

$$\beta \in X_{\infty,1}, \quad \text{and} \quad r_\zeta \in X_{\infty,1}, \quad \text{for any } \zeta > 0, \tag{3.2}$$

where $X_{\infty,1}$ is the dual space of $X_{0,1}$. It consists of measurable functions f for which

$$\|f\|_{\infty,1} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} \frac{|f(x)|}{1+x} < \infty.$$

The duality pairing is given by the integral

$$< f, g > = \int_0^\infty f(x)g(x)dx.$$

In this section, we make use of the theory of semilinear abstract Cauchy problems. The idea is to show that the linear operator induced by fragmentation, growth, and sinking of the aggregates generates a strongly continuous semigroup. This linear operator is then perturbed by the nonlinear operator induced by coagulation of the aggregates and yields the existence and uniqueness of a local solution to the evolution equation. Global existence is thereafter obtained by showing that the local solution does not blow up infinite time.

3.1. Analysis of the linear part

Let us denote by T_ζ the realization of $\mathcal{T}_\zeta$ (defined via (2.3)) on the domain

$$D_\zeta = \{\psi \in X_{0,1}; q\psi \in X_{0,1},\ r_\zeta\psi \in AC((x_0,\infty))\ (r_\zeta\psi)_x \in X_{0,1}\}, \tag{3.3}$$

if r_ζ^{-1} non-integrable at 0, and on the domain

$$D_{\beta,\zeta} = \left\{\psi \in D_\zeta : \lim_{x\to 0^+} r_\zeta(x)\psi(x) = \int_0^\infty \beta(y)\psi(y)dy\right\}, \tag{3.4}$$

otherwise. In addition, let B be the realization of $\mathcal{B}$ (defined via (2.4)) on the domain

$$D(B) = D(T_\zeta) = \{\psi \in X_{0,1}; q\psi \in X_{0,1},\ r\psi \in AC((0,\infty))\ (r\psi)_x \in X_{0,1}\}.$$

For further use, we define for any iron concentration $\zeta > 0$

$$R_\zeta(x) := \int_1^x \frac{1}{r_\zeta(s)}ds, \quad Q_\zeta(x) := \int_1^x \frac{q_\zeta(s)}{r_\zeta(s)}ds. \tag{3.5}$$

Theorem 3.1. *The operator* $(T_\zeta;\ D(T_\zeta))$ *with the resolvent given by*

$$(Res_\zeta(\lambda)f)(x) = \frac{e^{\lambda R_\zeta(x)+Q_\zeta(x)}}{r_\zeta(x)}\int_x^\infty \frac{e^{\lambda R_\zeta(y)+Q_\zeta(y)}}{r_\zeta(y)} f(y)dy, \tag{3.6}$$

for any $\lambda > 0$ *and* $f \in X_{0,1}$ *is the generator of a strongly continuous positive semigroup of contractions, say* $\{S_{T_\zeta}(t)\}_{t\geq 0}$ *on* $X_{0,1}$.

Proof. The case $r_\zeta > 0$ representing fragmentation with growth is similar to [17] and the case $r_\zeta < 0$ representing decay can be found in detail in [15]. In both cases, the expression of the resolvent is obtained and Hille-Yosida inequality is proven to be satisfied. □

Theorem 3.2. *There exists an extension* $(G_\zeta;\ D(G_\zeta))$ *of the operator* $(T_\zeta + B;\ D(T_\zeta))$, *which generates a positive strongly continuous semigroup* $(S_{G_\zeta}(t))_{t\geq 0}$ *in* $X_{0,1}$. *Moreover, the generator* G_ζ *is characterized by*

$$(\lambda I - G_\zeta)^{-1}\psi = \sum_{n=0}^\infty (\lambda I - T_\zeta)^{-1}[B(\lambda I - T_\zeta)^{-1}]^n\psi, \tag{3.7}$$

for $\psi \in X_{0,1}$ *and* $\lambda > 0$.

Proof. The proof is a generalization of a similar result for the space $X_{0,1}$, obtained in [4] by assuming that the fragmentation rate a is linearly bounded. The analysis in [4] can be easily extended to general fragmentation rates because the fragmentation equation behaves well in the bigger space $X_1 := L_1((0, \infty), xdx) = \{\phi; \int_0^\infty |\phi(x)| xdx < +\infty\}$. A complete proof of this theorem is available in [15]. □

Theorem 3.3. *Assume* $\lim_{x \to x_0} q_\zeta(x) = \lim_{x \to 0} a(x) + d(x) + s_\zeta(x) < +\infty$, *then the generator of the semigroup* $(S_{G_\zeta}(t))_{t \geq 0}$ *is given by*

$$G_\zeta = \overline{T_\zeta + B}.$$

Proof. The theory of extension of operator is instrumental in the proof of this theorem. In the case r^{-1} non-integrable at x_0, the assumption made is not necessary. The semigroup $(S_{G_\zeta}(t))_{t \geq 0}$ is honest for arbitrary fragmentation rate $a \in L_{\infty, loc}((0, \infty))$ and grazing rate $d \in L_\infty((0, \infty))$. The proof is analogous to the analysis for honesty performed in [3]. For r^{-1} integrable at 0, the proof is obtained in a similar way as in [4], where honesty was investigated in the space $X_{0,1}$.

□

3.2. Global solutions of the transport equation with fragmentation and coagulation

In this section, we show the existence of a global solution to the full evolution problem (2.1) endowed with its initial and boundary conditions. This evolution equation is represented by the following semilinear abstract Cauchy problem:

$$\frac{du_\zeta}{dt}(t) = [G_\zeta + K]u_\zeta(t),$$

$$u_\zeta(0) = u_0, \tag{3.8}$$

where K is the realization of the expression

$$[\mathcal{K}\psi](x) = \frac{1}{2}\int_0^x k(x-y,\, y)\psi(x-y)\psi(y)dy - \psi(x)\int_0^\infty k(x,\, y)\psi(y)dy, \quad (3.9)$$

for nonzero ψ on the space $X_{0,1}$ and $K(0) = 0$. Since the linear semigroups $(S_{G_\zeta}(t))_{t\geq 0}$ is positive, we shall work in the positive cone of $X_{0,1}$, denoted by $X_{0,1}^{+}$.

Theorem 3.4. *Let* $u_0 \in X_{0,1}^{+}$, *then the Cauchy problem*

$$\frac{du_\zeta}{dt}(t) = G_\zeta[u(t)] + K[u(t)], \quad u_\zeta(0) = u_0, \quad (3.10)$$

has a unique global solution.

Proof. In order to prove that (3.8) has a solution, which is global in time, we shall proceed in a usual way [4] by converting it to the integral equation

$$u_\zeta(t) = S_G(t)u_0 + \int_0^t S_G(t-s)K[u_\zeta(s)]ds, \quad t \geq 0, \quad (3.11)$$

where $(S_G(t))_{t\geq 0}$ is the semigroup generated by G. We use the fact that $X_{0,1+}$ is a complete metric space as a closed subspace of a Banach space, see [18, Theorem 6.1.2]. The method is analogous to the proof of global existence in [4] with the space $X_{0,1}$. □

4. Numerical Simulations

This section provides a prediction of the phytoplankton biomass

$$N_\zeta(t) = \int_0^\infty x u_\zeta(t,\, x)dx,$$

from the year 2010 to 2030. A numerical method shall be used and numerical simulations performed over the evolution equation

$$\frac{\partial}{\partial t} u_{\zeta}(t, x) = -\frac{\partial}{\partial x}[r_{\zeta}(x)u(t, x)] - s_{\zeta}(x)u(t, x) - d(x)u(t, x)$$

$$- a(x)u(t, x) + \int_{x}^{\infty} a(y)b(x|y)u(t, y)dy$$

$$- u(t, x)\int_{0}^{\infty} k(x, y)u(t, y)dy$$

$$+ \frac{1}{2}\int_{0}^{x} k(x - y, y)u(t, x - y)u(t, y)dy,$$

describing the dynamics of the phytoplankton population.

4.1. Empirical data and estimation of kernels

We shall set the initial conditions to be $u(x, 0) = 10^{8}\chi e^{-x}$, where χ is a positive real number. To determine the effects of ocean iron concentration ζ on the dynamics of the population, we follow [20] and make use of Runge-Kutta methods extended with a quadrature technique (Pouzet type) in order to simulate the corresponding non-dimensionalized evolution equation. We investigate the dynamics of the plankton population in the HNLC regions and predict the evolution corresponding to some specified values of iron concentration. The kernels used in the simulations are summarized in the following table:

Table 1. Kernels used in the simulations

Description	Kernels
Fragmentation rate	$a(x) = x^{\alpha}$
Daughter particles distribution rate	$b(x, y) = \frac{(\nu + 2)x^{\nu}}{y^{\nu+1}}$
Grazing rate	$d(x) = d$
Aggregation rate	$k(x, y) = k$
Growth rate	$r_{\zeta}(x) = (\zeta - \zeta_c)x$
Sinking rate	$s_{\zeta}(x) = \zeta x$
Renewal rate	$\beta_{\zeta}(x) = \zeta x$

Parameter values were estimated from available experimental information. In the events where no observational data could be obtained, parameter values were picked out to provide the best qualitative numerical simulation results. This is in line with previous articles successfully simulating the dynamics of phytoplankton [1, 12].

Table 2. Parameter values used in the simulations

Parameter	Value	Source
ζ_c	2	[1, 12]
ζ	(0.02,10)	fitted
d	20	fitted
k	500	fitted
ν	0	[12, 15]
χ	(1.8, 2.2)	fitted
α	0.75	[1, 12, 15]

4.2. Computational simulations

The critical iron concentration value that determines the growth or decay of phytoplankton aggregates is approximately 2 nanomole (nM) per litre in the high nitrate low chlorophyll (HNLC) regions of the oceans [15]. Simulations are performed for iron concentration values around this critical value

$$\zeta \in \{0.02, 0.1, 0.5, 1, 1.4, 1.6, 2.1, 2.4, 2.5, 3, 5, 10\},$$

and arbitrary χ values in the range (1.8, 2.2). The fragmentation daughter particle distribution kernel is chosen to be binary $\nu = 0$ and the fragmentation rate $a(x) = x^{0.75}$ is chosen to be linearly bounded. The coalescence rate k and the grazing rate d are taken to be 500 and 20, respectively. The simulation results are summarized in Figures 1, 2, and 3, respectively.

4.3. Interpretation of the results and discussions

The simulations results suggest that iron (Fe) availability is the primary factor controlling phytoplankton production in HNLC regions of the oceans. The population biomass is seen to increase with $\zeta > \zeta_c$ and decrease otherwise over a long period of time. The increment in the biomass observed for $\zeta = 1.4, 1.6 < \zeta_c$ in the first 10 year interval can be explained by the fact that HNLC regions are in general nutrient-rich areas and plankton population may grow with a very little amount of iron available $(\zeta < 0.03\text{nM})$. However, the population reduces substantially in the long run because the acquisition of sufficient iron (Fe) for phytoplankton syntheses of Chl and nitrate reductase needed by them to use the abundant major nutrients becomes a serious problem [10, 12]. Another important feature in Figure 2 is the suggestion that maintaining the level of iron concentration just a little bit above the critical value ζ_c has the potential to ensure a long term satisfactory level of phytoplankton biomass. It also clearly shows that raising the level of marine iron too much above ζ_c does not present any technical advantage in the long run. The results of this study indicate that iron hypothesis can be implemented in a very cost-effective way and produce impressive results. In summary, we have made used of mathematical and computational techniques in order to present a very efficient method to increase the world phytoplankton biomass. We recommend this method to be used in future in order to systematically reduce the effects of global warming.

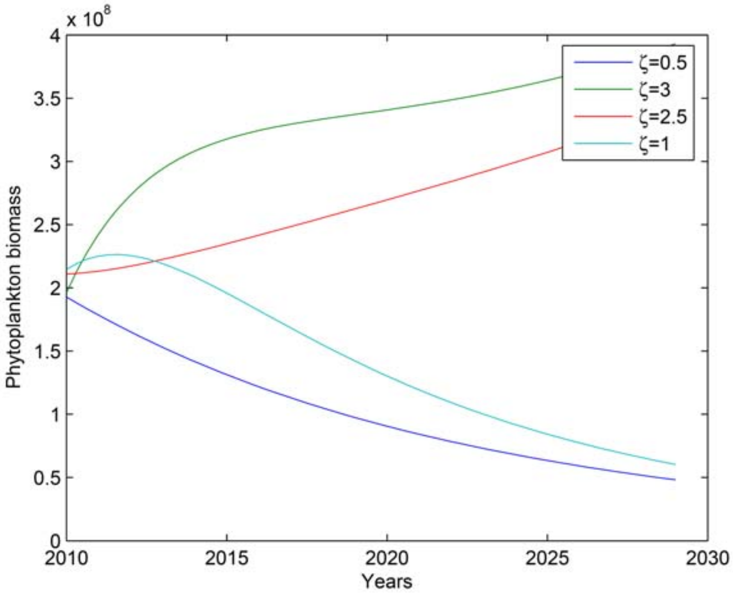

Figure 1. Comprehensive prediction of the phytoplankton biomass.

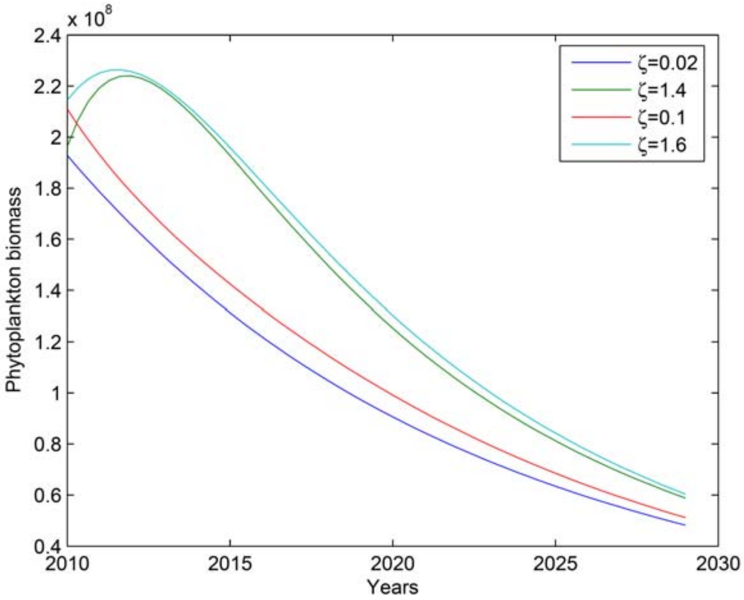

Figure 2. Prediction of the phytoplankton biomass with $\zeta < \zeta_c$.

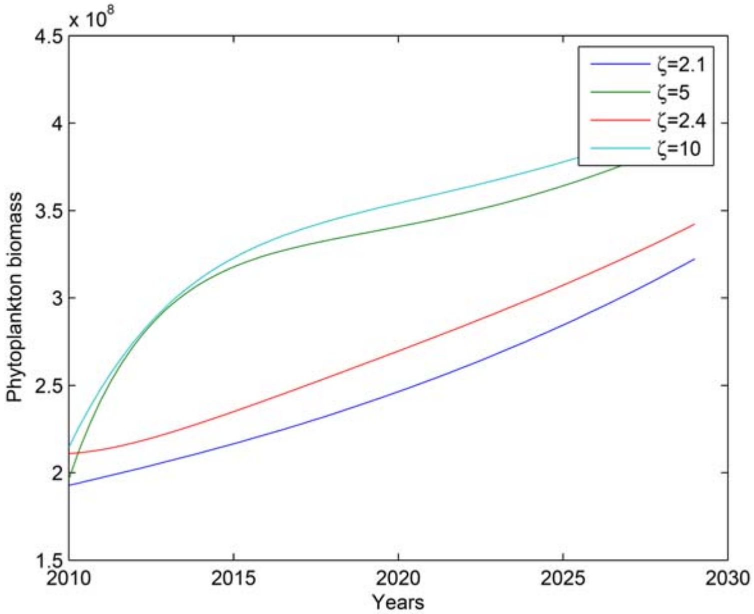

Figure 3. Prediction of the phytoplankton biomass with $\zeta > \zeta_c$.

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